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HOPF BIFURCATION AND ATTRACTIVITY

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Hopf bifurcation and attractivity

by

S. van Gils

ABSTRACT

In this paper the problem of attractivity of bifurcating orbits in Hopf Bifurcation is reduced to the analysis of linear algebraic systems. Explicit formulas in the case of two and three dimensions are given. They are applied to the Lorenz equations.

KEY WORDS & PHRASES: *Hopf bifurcation, attractivity, Lyapunov function, Lorenz equations*

INTRODUCTION

Consider a one parameter family of O.D.E.'s

$$\begin{aligned} &\dot{x} = f(x,\mu) \\ &x \in \mathbb{R}^n, \quad \mu \in \mathbb{R} \\ &f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \text{ analytic} \\ &f(0,\mu) \equiv 0. \end{aligned} \tag{0.1}$$

Denote $f(x,0)$ by $f_0(x)$ and assume that the Jacobian matrix $f_x(0,\mu)$ has a conjugate pair of simple eigenvalues, $\alpha(\mu) \pm i\beta(\mu)$ with $\alpha(0) = 0$, $\dot{\alpha}(0) > 0$ and $\beta(0) \neq 0$. We may regard (0.1) as a perturbation of the differential equation

$$\dot{x} = f_0(x) \tag{0.2}$$

to which we refer as the unperturbed differential equation. In physical systems the value of μ may correspond to the value of an inductance, a resistance or a spring constant. See for example the tuned grid oscillator in Chapter I. In chemical systems it may correspond to a side condition as in the example of the *Brusselator* [13]. It was shown by HOPF [6] that there are periodic orbits bifurcating from the zero solution. For sufficiently small μ the periodic solutions generally exist only for $\mu > 0$ or for $\mu < 0$; it's also possible that they exist only for $\mu = 0$. If one plots the amplitude-parameter graph the three situations of fig. 1 are all possible and of physical interest.

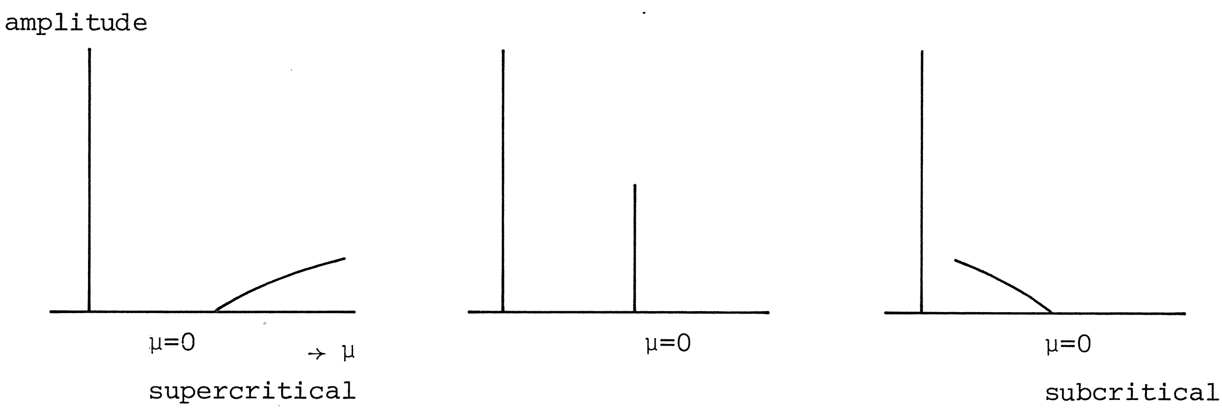

fig. 1

We can demonstrate it by the following prototype example.

Consider the following system of differential equations

$$\begin{cases} \dot{x} = \mu x + y + cx(x^2+y^2) \\ \dot{y} = -x + \mu y + cy(x^2+y^2) \end{cases} \qquad (0.3)$$

which system is in polar coordinates equivalent to

$$\begin{cases} \dot{r} = \mu r + cr^3 \\ \dot{\theta} = 1 \end{cases} \qquad (0.4)$$

where c is a fixed but arbitrarily chosen constant and μ is the bifurcation parameter.

a) If $c < 0$ then there is a periodic solution of (0.4) for $\mu > 0$

$$\begin{cases} r = \sqrt{-\frac{\mu}{c}} \\ \theta(t) = t+c \end{cases} . \qquad (0.5)$$

In this case the periodic solution is stable, the origin is asymptotically stable at $\mu = 0$. If we define $\varepsilon = \sqrt{-\frac{\mu}{c}}$ then $\mu(\varepsilon) = -c\varepsilon^2$ (supercritical bifurcation).

b) If $c > 0$ then there is a periodic solution of (0.4) for $\mu < 0$

$$\begin{cases} r = \sqrt{-\frac{\mu}{c}} \\ \theta(t) = t+c \end{cases} . \qquad (0.6)$$

In this case the periodic solution is unstable, the origin is unstable at $\mu = 0$. Let $\varepsilon = \sqrt{-\frac{\mu}{c}}$ then $\mu(\varepsilon) = -c\varepsilon^2$ (subcritical bifurcation).

c) The case $c = 0$ yields the second case displaced in figure 1.

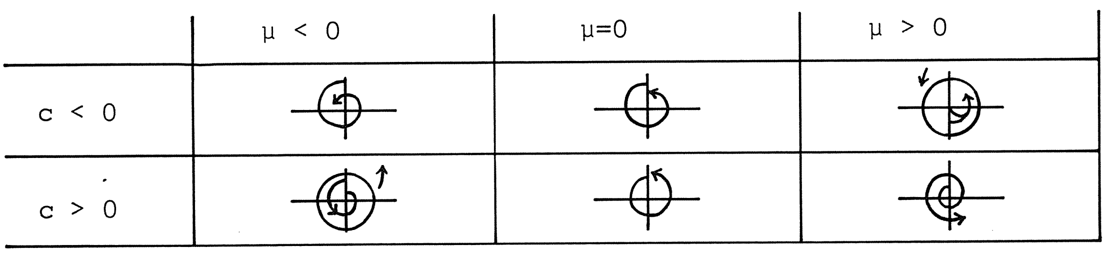

fig. 2

There are three problems involved with Hopf bifurcation

(i) the existence of the bifurcation

(ii) the stability or instability of the periodic orbit and

(iii) the stability of the origin when $\mu = 0$.

These three problems are related to each other in the following way. If the conditions for the existence of Hopf Bifurcation, which we already stated below formula (0.1) are fulfilled, then the following is true.

If the origin is asymptotically stable there; occurs a supercritical bifurcation and the periodic orbits are unstable if the origin is unstable when $\mu = 0$ there occurs a subcritical bifurcation and the periodic orbits are stable. This will be shown in detail in Chapter II for two dimensional systems and for higher order systems we will state a theorem of Chafee which, together with Hopf's theorem, proves the relation.

This relation enables us to reduce the problem whether the bifurcating periodic solution is stable or not to the analysis of algebraic systems. In chapter III we will derive a stability formula for two dimensional systems and we will give some examples. In chapter IV we do the preparations in order to derive a formula for three dimensional systems and we will derive that formula in chapter V. In chapter VI we deal with the Lorenz equations.

The Lorenz equations form a system of three ordinary differential equations which, in a simplified way, describe the flow occurring in a layer of uniform depth when the temperature difference between the upper and lower surfaces is maintained at a constant value.

We will apply the formula, derived in chapter V to this system. It will turn out that our results are in contradiction with those of MARSDEN and McCRACKEN ([12], pag 145-148).

I. AN INTRODUCTORY EXAMPLE

I.1 SELF OSCILLATIONS OF A TUNED GRID OSCILLATOR

Consider a vacuum tube oscillator with a tuned grid circuit (fig. 3, [2] chapter IX). The Kirchof equation determining the current in the oscillating circuit of the tube gives

$$L\dot{i} + Ri + \frac{1}{C}\int idt = M\dot{I}_a \qquad (\dot{\ }) = \frac{d}{dt}. \tag{I.1}$$

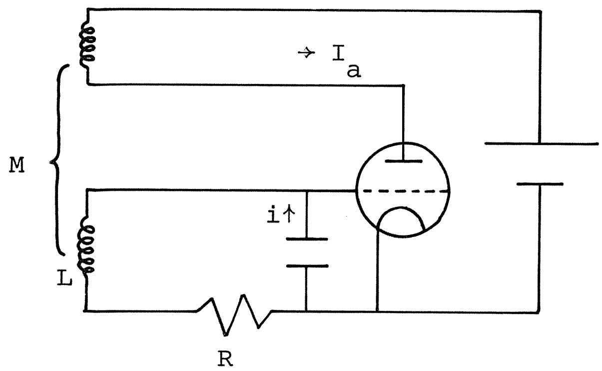

fig. 3

We assume that I_a is given by

$$I_a = V_s(\alpha_1 v + \beta_1 v^2 - \gamma_1 v^3) \qquad \gamma > 0, \tag{I.2}$$

where $v = V_g/V_s$, V_s is the saturation voltage, V_g is the grid voltage. Since $V_g = \frac{1}{C}\int idt$ and $v = \frac{1}{CV_s}\int idt$ we can write (I.1) in the form

$$LC\ddot{v} + RC\dot{v} + v = \frac{M}{V_s}\dot{I}_a. \tag{I.3}$$

Using (I.2) and making some elementary transformations and writing t for $\omega_0 t$ we obtain:

$$\ddot{v} + v = (\alpha + 2\beta v - 3\gamma v^2)\dot{v}. \tag{I.4}$$

Here $\omega_0^2 = \frac{1}{LC}$, $\alpha = (M\alpha_1 - RC)\omega_0$, $\beta = M\beta_1\omega_0$, $\gamma = M\gamma_1\omega_0$ and α, β, γ are dimensionaless parameters.

Scaling α, β, γ by $\alpha \to \mu\alpha$, $\beta \to \mu\beta$, $\gamma \to \mu\gamma$ we have

$$\ddot{v} + v = \mu(\alpha + 2\beta v - 3\gamma v^2)\dot{v}. \tag{I.5}$$

At $\mu = 0$ we have

$$\ddot{v} + v = 0 \tag{I.6}$$

for which the general solution is

$$v(t) = a\cos(t+\phi) \qquad \alpha,\ \phi \in \mathbb{R}. \tag{I.7}$$

For (I.4) we seek a periodic solution $v(t,\mu)$ with angular frequency $\Omega(\mu)$ such that $\Omega(0) = 1$ and $v(t,0) = a\cos(t+\phi)$. We call $a\cos(t+\phi)$ a generating solution for (I.4) and we say that $v(t,\mu)$ bifurcates from that generating solution. We make the change of variables

$$\tau = \Omega(\mu)t$$

$$z(\tau,\mu) = v(t,\mu)$$

so that $z(\tau,\mu)$ is to have period 2π in τ. We shall seek $z(\tau,\mu)$ and $\Omega(\mu)$ as power series in μ

$$\left\{\begin{array}{lll} z(\tau,\mu) & = z_0(\tau) + z_1(\tau)\mu + z_2(\tau)\mu^2 + \dots \\ \Omega(\mu) & = 1 + \Omega_1\,\mu + \Omega_2\,\mu^2 + \dots \end{array}\right\}; \tag{I.8}$$

moreover we impose on $z_n(\tau)$ the boundary condition

$$\dot{z}_n(0) = 0 \qquad (n = 0,1,\dots). \tag{I.9}$$

Substituting (I.8) into (I.5) and collecting like powers of μ we obtain a recursive scheme of ordinary differential equations. Collecting the terms in μ^0 we obtain

$$\ddot{z}_0(\tau) + z_0(\tau) = 0, \tag{I.10}$$

which gives us together with (I.9) the solution

$$z_0(\tau) = a_0 \cos\tau \qquad a_0 \in \mathbb{R}. \tag{I.11}$$

Collecting the terms in u^1 we obtain

$$\ddot{z}_1 + z_1 = f_1 + 2\Omega_1\, a_0 \cos\tau, \tag{I.12}$$

where

$$f_1 = (\alpha + 2\beta a_0 \cos\tau - 3\gamma\, a_0^2 \cos^2\tau)(-a_0 \sin\tau),$$

(I.12) with boundary condition $\dot{z}_1(0) = 0$ has a unique solution iff

$$\int_0^{2\pi} (f_1(\xi) + 2\Omega_1\, a_0 \cos\xi)\cos\xi\, d\xi = 0, \qquad \text{(I13.a)}$$

and

$$\int_0^{2\pi} (f_1(\xi) + 2\Omega_1\, a_0 \cos\xi)\sin\xi\, d\xi = 0. \qquad \text{(I13.b)}$$

$$\left.\begin{array}{ll} \text{(I.10a)} \Rightarrow & 2\pi\, \Omega_1 a_0 = 0 \\ \text{(I.10b)} \Rightarrow & -\alpha a_0 \pi + \tfrac{3}{4}\pi\, \gamma\, a_0^3 = 0 \end{array}\right\} \Rightarrow$$

(i) $\Omega_1 = 0$

(ii) $a_0 = \begin{cases} 0 & \text{if } \alpha \le 0 \\ \sqrt{\dfrac{4\alpha}{3\gamma}} & \text{if } \alpha > 0 \end{cases}.$

With a more rigorous method like that of Lyapunov-Schmidt ([15], chapter IV) one can show that if $\alpha > 0$ (I.5) has a nontrivial periodic solution

$$\begin{cases} v(t,\mu) = \sqrt{\dfrac{4\alpha}{3\gamma}} \cos\Omega(\mu)t + O(\mu) \\ \Omega(\mu) \;\;= 1 + O(\mu^2) \end{cases}. \qquad \text{(I.14)}$$

If α varies continuously from negative to positive values, oscillations start at $\alpha = 0$, their amplitude increasing continuously. When $\alpha < 0$ the system behaves like a damped oscillator.

We can write (I.4) as a system of ordinary differential equations:

$$\left\{\begin{array}{l} \dot{x} = y \\ \dot{y} = -x + \alpha y + 2\beta xy - 3\gamma x^2 y \end{array}\right\}. \qquad \text{(I.15)}$$

The linearized system is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{(I.16)}$$

$$= A_\alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

A_α has one pair of complex conjugate eigenvalues $\lambda(\alpha) = \frac{1}{2}\alpha \pm \frac{1}{2}i\sqrt{4-\alpha^2}$ which implies that

$$\mathrm{Re}\,\dot{\lambda}(0) = \tfrac{1}{2}, \quad \mathrm{Re}\,\lambda(0) = 0 \quad \text{and} \quad \mathrm{Im}\,\lambda(0) = 1.$$

This is a special case of a more general situation for which HOPF [6] has derived a very useful theorem which will be given in the next section.

I.2 WHAT IS HOPF BIFURCATION

Hopf bifurcation is the bifurcation of a periodic orbit from the origin in the following situation. Consider a one parameter family of O.D.E.'s

$$\dot{x}(t) = f(x(t),\mu)$$
$$x \in \mathbb{R}^n, \quad \mu \in \mathbb{R} \qquad \text{(I.17)}$$

(H1) $f(0,\mu) = 0$

(H2) $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, f analytic

(H3) $f_x(0,\mu)$ has exactly one simple pair of complex conjugate eigenvalues $\lambda(\mu)$ and $\overline{\lambda(\mu)}$ such that $\mathrm{Re}\,\lambda(0) = 0$, $\mathrm{Re}\,\dot{\lambda}(0) > 0$ and $\mathrm{Im}\,\lambda(0) \neq 0$

(H4) At $\mu = 0$, the other eigenvalues have negative real parts.

<u>THEOREM 1</u> Hopf 1942. *Let* (I.17) *satisfy* (H1), (H2) *and* (H3). *Then there exist unique functions* $x = x(t,\varepsilon)$, $\mu = \mu(\varepsilon)$ *and* $T = T(\varepsilon)$ *defined and analytic for* ε *sufficiently small satisfying*

$$x(t,0) \equiv 0, \quad \mu(0) = 0 \quad \textit{and} \quad T(0) = 2\pi/\mathrm{Im}\,\lambda(0)$$

and having the property that for each ε *sufficiently small and not zero*

$x(t,\varepsilon)$ *is a real non-zero* T-*periodic solution of* (I.17) *with* $\mu = \mu(\varepsilon)$ *and* $T = T(\mu(\varepsilon))$.

For arbitrary large L *there are two positive numbers* a *and* b *such that for* $|\mu| < b$ *there exist no periodic solution besides the stationary solution and the solutions* $x(t,\varepsilon)$ *whose period is smaller than* L *and which lies entirely in* $B(0,a) = \{x \in \mathbb{R}^n \mid |x| < a\}$. *For sufficiently small* μ *the periodic solutions generally exist only for* $\mu > 0$ *or for* $\mu < 0$; *it is also possible that they exist only for* $\mu = 0$.

The last paragraph will be more fully explained later in the thesis.

II. HOW THE BEHAVIOUR AT $\mu = 0$ INFLUENCES STABILITY AND BIFURCATION

A. Throughout this part we assume $x \in \mathbb{R}^2$. Let (I.17) satisfy (H1), (H2) and (H3). Then we can write this system as

$$\begin{cases} \dot{x} = \alpha(\mu)x - \beta(\mu)y + \phi(x,y) \\ \dot{y} = \beta(\mu)x + \alpha(\mu)y + \phi(x,y) \end{cases}$$

$$\alpha(0) = 0; \quad \dot{\alpha}(0) > 0; \quad \beta(0) > 0 \tag{II.1}$$

$$\phi,\psi \text{ analytic}; \quad \phi,\psi \in o(x^2+y^2),\ (x,y) \to (0,0).$$

<u>THEOREM 2A</u>: *If at* $\mu = 0$ *the origin is attractive, then there exists an analytic function* $V = V(x,y)$ *such that* $\dot{V}_{II.1} < 0$ *and* $V > 0$ *in a neighbourhood of the origin.*

<u>THEOREM 3A</u>: *If at* $\mu = 0$ *the origin is attractive, then*

(i) *bifurcation takes place for* $\mu > 0$

(ii) $\exists \mu^* > 0$ *such that for* $0 < \mu < \mu^*$ *the origin is unstable and the bifurcating periodic solution is attractive.*

<u>PROOF OF THM 2A</u>: denote $\beta(0)$ by λ; when $\mu = 0$ we can write (II.1) as

$$\begin{cases} \dot{x} = -\lambda y + \sum x_k(x,y) \\ \dot{y} = \lambda x + \sum y_k(x,y) \end{cases} \tag{II.2}$$

where X_k and Y_k are homogeneous polynomials of degree k. We try to find a function $V = V(x,y)$ such that $\dot{V}_{II.2} < 0$ in a neighbourhood of the origin. Let

$$V(x,y) = x^2+y^2 + \sum_{3}^{\infty} f_i(x,y), \tag{II.3}$$

where f_i stands for a homogeneous polynomial of degree i. We determine these polynomials in a suitable way.

$$\dot{V}_{II.2} = (2x + \frac{\partial f_3}{\partial x} + \ldots)(-\lambda y + X_2 + X_3 + \ldots\ )+ \\ (2y + \frac{\partial f_3}{\partial y} + \ldots)(\ \lambda x + Y_2 + Y_3 + \ldots\ ). \tag{II.4}$$

If we denote by V_m the terms in (II.4) of m-th order in x and y we conclude that

$$V_m = \lambda(x \frac{\partial f_m}{\partial y} - y \frac{\partial f_m}{\partial x}) + F_m(x,y), \tag{II.5}$$

where F_m is a homogeneous polynomial in x and y having degree m. The coefficients of F_m are determined by $f_3, f_4, \ldots, f_{m-1}$, $X_2, X_3, \ldots, X_{m-1}$, $Y_2, Y_3, \ldots, Y_{m-1}$.

Now we will show that

(i) if m is odd there is a uniqe f_m such that $V_m = 0$

(ii) if m is even there is a unique G_m such that the equation $V_m = G_m(x^2+y^2)^{m/2}$ has a solution.

In order to proof this set

$$f_m = a_1x^m + a_2x^{m-1}y + \ldots + a_{m+1}y^m \tag{II.6}$$

and

$$F_m = -c_1x^m - c_2x^{m-1}y - \ldots - c_{m+1}y^m . \tag{II.7}$$

Solving $V_m = 0$ is equivalent to solving

$$ja_{j+1} - (m-j+2)a_{j-1} = \frac{1}{\lambda} c_j \qquad j = 1 \ldots\ldots m+1. \tag{II.8}$$

$$a_{m+2} = a_0 = 0$$

The first assertion follows from corollary 2 in the appendix.

If m is even it follows from corollary 3 in the appendix that the rank of the coefficient matrix of (II.8) equals m−1. So there are numbers

$$M_1,\dots,M_n$$

such that $V_m = 0$ is solvable iff

$$M_1c_1 + \dots + M_{m+1}c_{m+1} = 0. \tag{II.9}$$

In general this will not be the case, but if

$$(x^2+y^2)^{m/2} = d_1x^m + \dots + d_{m+1}y^m \tag{II.10}$$

there is a unique number G_m such that

$$\sum_{i=1}^{m+1} M_i(\frac{1}{\lambda}c_i + G_m d_i) = 0. \tag{II.11}$$

Hence $V_m = G_m(x^2+y^2)^{m/2}$ has a solution.

Without computing the numbers d_i and M_i we are able to compute the number G_m. Integrating (II.5) over the unit circle and remembering that $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$ we see that

$$G_m = \frac{1}{2\pi}\int_0^{2\pi} F_m(\cos\theta, \sin\theta)d\theta. \tag{II.12}$$

Now we state the following lemma.

<u>LEMMA</u>.

(i) *if not all* $G_{2N} = 0$ (N = 1,2,..,) *let* M *denote the first number such that* $G_M \neq 0$,
if $G_M < 0$, *then the origin is asymptotically stable,*
if $G_M > 0$, *then the origin is unstable.*

(ii) *if all* $G_{2N} = 0$, *then the origin is stable but not asymptotically stable.*

<u>PROOF.</u>

(i) consider

$$V^M(x,y) = x^2+y^2 + f_3(x,y) + \ldots + f_M(x,y) \qquad \text{(II.13)}$$

$$\dot{V}^M_{II.2} = G_M(x^2+y^2)^{M/2} + \chi(x,y)$$

where

$$\chi(x,y) = o((x^2+y^2)^{M/2}) \qquad (x,y) \to (0,0).$$

If $G_M > 0$ then $\exists\delta,\ \eta > 0$ such that

$$|x^2+y^2| < \delta \Rightarrow \dot{V}^M_{II.2} > \eta$$

(ii) in this case one can prove [9] that V(x,y) is analytic in a neighbourhood of the origin and that V(x,y) is a first integral of (II.2). From this the result follows.

If the origin is attractive there must be an M such that $G_M < 0$ and V(x,y) is given by (II.13). This proves theorem 2A.

PROOF OF THEOREM 3A:

(i) if $\mu < 0$, then both eigenvalues have negative real parts, so then the origin is attractive. From theorem 1 we know that bifurcation takes place and as the origin is attractive at $\mu = 0$ bifurcation will occur for $\mu > 0$.

(ii) if $\mu > 0$, then both eigenvalues have positive real parts, so then the origin is unstable. From theorem 1 it follows that $\exists U_0 \ni 0$, $\exists \mu_1$, such that $\exists!$ periodic solution $x(t,\mu)$, $(0 < \mu < \mu_1)$, in U_0 with $0 < T(\mu) < \infty$. From the previous lemma it follows that $\exists U_1 \subset U_0$ which includes the origin and $\exists\ V^m(x,y)$ such that $\dot{V}^M_{II.2} < 0$ in U_1. This implies the existence of a constant C such that the equation $V^M(x,y) = C$ implicitly defines an arc ℓ without contact for (II.2). Consequently there is a μ_2 such that $|\mu| < \mu_2$ implies that ℓ is an arc without contact for (II.1).

Take $\mu^* = \min\{\mu_1,\mu_2\}$ and U_1 so small that the origin is the only equilibrium point of (II.1) for $|\mu| < \mu^*$. Then it follows from the Poincaré-Bendixson Thereom that the periodic orbits which lie inside ℓ are attractive.

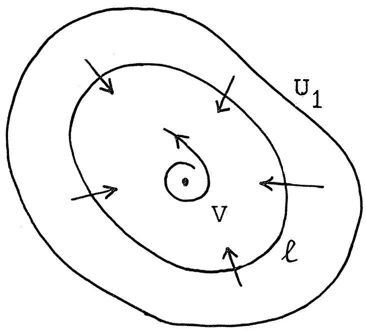

fig. 4

B. In this part we remove the condition $x \in \mathbb{R}^2$

For the higher dimensional case it is quite difficult to prove the analogue of theorem 3A, but the desired result is contained in the Hopf Theorem in $\mathbb{R}^n$ ([12], pg. 81). So we will state the analogue without proof

<u>THEOREM 4B</u>. *Let* (I.17) *satisfy* $(H_1)(H_2)(H_3)$ *and* (H_4) *and let the origin be attractive at* $\mu = 0$. *Then there exist unique functions* $x = x(t,\varepsilon)$, $\mu = \mu(\varepsilon)$ *and* $T = T(\varepsilon)$ *defined and analytic for* ε *sufficiently small satisfying*

$$x(t,0) \equiv 0, \quad \mu(0) = 0 \quad \text{and} \quad T(0) = 2\ /I_m\ \lambda(0)$$

and having the property that for each ε *sufficiently small and not zero* $x(t,\varepsilon)$ *is a real non-zero* T*-periodic solutions of* (I.17) *with* $\mu = \mu(\varepsilon)$ *and* $T = T(\mu(\varepsilon))$.

Furthermore the orbits exists for $\mu > 0$ and are attractive in a full neighbourhood in $\mathbb{R}^n$.

At this stage I want to make the following remarks. Theorem 4b, that is taken from the book of Marsden & McCracken tells us that the bifurcating orbits are attractive in a full neighbourhood in $\mathbb{R}^n$, but they give no rigorous proof.

A theorem of CHAFEE [3] shows that the closed orbits lie on a local integral manifold, homeomorphic to an open disk in $\mathbb{R}^2$ and containing the origin $x = 0$. Moreover the shows that within this local manifold the orbits are attractive. Assumptions H_4 guaranties that the manifold itself is attracting. And recently a. Tesei proved rigorously that these two facts imply the attractivity of the bifurcating orbits in a full neighbourhood.

III. A STABILITY FORMULA FOR TWO DIMENSIONAL SYSTEMS

In order to apply the previous theorems it is necessary to calculate G_4. Denote the coefficient of $x^i y^j$ in the first equation of (II.2) by X_{ij} and by Y_{ij} for the second equation.

From the definition of V_3 (above II.5) we see that

$$F_3 = 2(xX_2+yY_2). \qquad \text{(III.1)}$$

Now we set

$$f_3 = a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \qquad \text{(III.2)}$$

and we require that $V_3 = 0$. This gives the following conditions on the coefficients of f_3

$$\left\{\begin{array}{rrrrl} & \lambda a_2 & & & = -2X_{20} \\ -3\lambda a_1 & + & 2\lambda a_3 & & = -2X_{11} - 2Y_{20} \\ & -2\lambda a_2 & + & 3\lambda a_4 & = -2X_{02} - 2Y_{11} \\ & & -\lambda a_3 & & = -2Y_{02} \end{array}\right\} \qquad \text{(III.3)}$$

We conclude that

$$f_3(x,y) = \frac{2X_{11}+2Y_{20}+4Y_{02}}{3\lambda}x^3 - \frac{2X_{20}}{\lambda}x^2y + \frac{2Y_{02}}{\lambda}xy^2 - \frac{4X_{20}+2X_{02}+2Y_{11}}{3\lambda}y^3 \qquad \text{(III.4)}$$

Evaluating (II.12) with m = 4 we get

$$G_4 = \frac{1}{2\pi}\int_0^{2\pi} F_4(\cos\theta, \sin\theta)d\theta, \qquad \text{(III.5)}$$

where F_4 is given by

$$F_4 = 2xX_3 + \frac{\partial f_3}{\partial x} X_2 + 2yY_3 + \frac{\partial f_3}{\partial y} Y_2 \qquad \text{(III.6)}$$

$$\frac{\partial f_3}{\partial x} X_2 = \left(\frac{2X_{11}+2Y_{20}+4Y_{02}}{\lambda}\right)(X_{20}x^4 + X_{11}x^3y + X_{02}x^2y^2)$$

$$- \left(\frac{4X_{20}}{\lambda}\right)(X_{20}x^3y + X_{11}x^2y^2 + X_{02}xy^3)$$

$$+ \left(\frac{2Y_{02}}{\lambda}\right)(X_{20}x^2y^2 + X_{11}xy^3 + X_{02}y^4), \qquad \text{(III.7)}$$

$$\frac{\partial f_3}{\partial y} Y_2 = \left(\frac{-2X_{20}}{\lambda}\right)(Y_{20}x^4 + Y_{11}x^3y + Y_{02}x^2y^2)$$

$$+ \left(\frac{4Y_{02}}{\lambda}\right)(Y_{20}x^3y + Y_{11}x^2y^2 + Y_{02}xy^3)$$

$$- \left(\frac{2X_{02}+2Y_{11}+4X_{20}}{\lambda}\right)(Y_{20}x^2y^2 + Y_{11}xy^3 + Y_{02}y^4). \qquad \text{(III.8)}$$

The final result is

$$G_4 = 1/4\{3X_{30} + X_{12} + Y_{21} + 3Y_{03} + \lambda^{-1}[2Y_{02}X_{02} - 2X_{20}Y_{20} + X_{11}(X_{02}+X_{20}) - Y_{11}(Y_{20}+Y_{02})]\} \qquad \text{(III.9)}$$

Compare [12] page 126, [1] page 253.

Applications

(i) tuned grid oscilator

We return to the example of Chapter I. The system is governed by the equations of (I.15). We saw that if we vary α continuously from negative to positive values, a periodic solution will bifurcate from the origin. Applying (III.9) to (I.15) with $\alpha = 0$ and remembering that $\gamma > 0$ we see that

$$G_{4(t.g.o)} = \frac{-3\gamma}{4} < 0.$$

Applying theorem 3A we know that the periodic solution is attractive.

(ii) brussalator [13]

$$\left\{\begin{array}{l}\dot{x} = a - (b+1)x + x^2y \\ \dot{y} = bx - x^2y\end{array}\right\} \quad a > 0. \qquad \text{(III.10)}$$

This is a mathematical model for a chemical reaction.

$\dot{x} = \dot{y} = 0 \Rightarrow x = a,\ y = ba^{-1}$. After transformation of the equilibrium point to the origin we get

$$\left\{\begin{array}{l}\dot{x} = (b-1)x + a^2y + ba^{-1}x^2 + 2axy + x^2y \\ \dot{y} = -bx \quad - a^2y - ba^{-1}x^2 - 2axy - x^2y\end{array}\right\}. \qquad \text{(III.11)}$$

The linearized system is given by

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}b-1 & a^2\\ -b & -a^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} := A_{a,b}\begin{pmatrix}x\\ y\end{pmatrix}. \qquad \text{(III.12)}$$

$A_{a,b}$ has two purely imaginary eigenvalues $\lambda = \pm ai$ iff $b = a^2+1$, and these eigenvalues cross the imaginary axis with speed $\frac{1}{2} > 0$. (Regard b as the bifurcation parameter and a as fixed).

Col $(a,-a+i)$ is an eigenvector of A_{a,a^2+1}, with eigenvalue ai.

Now use as transformation matrix in $\mathbb{R}^2$ the matrix

$$\begin{pmatrix}a & 0\\ -a & 1\end{pmatrix}. \qquad \text{(III.13)}$$

In the new base (III.11) with $b = a^2+1$ becomes

$$\left\{\begin{array}{l}\dot{x} = ay + (-a^2+1)x^2 + 2axy + ax^2y - a^2x^3 \\ \dot{y} = -ax\end{array}\right\} \qquad \text{(III.14)}$$

so

$$G_{4bruss} = -\frac{3}{4}a^2 - \frac{1}{4a}(2a(1-a^2)) = -\frac{1}{4}(a^2+2) \qquad \text{(III.14)}$$

We conclude that in the case of the brussalator, there is a neighbourhood of the origin in which the bifurcating periodic orbits are asymptotically stable.

IV. HOW TO HANDLE A THIRD ORDER SYSTEM

Let (I.17) satisfy (H_1) through (H_4) and let $x \in \mathbb{R}^3$. At $\mu = 0$ this system can be written as

$$\begin{cases} \dot{x} = -\lambda y + X(x,y,z) \\ \dot{y} = \lambda x + Y(x,y,z) \\ \dot{z} = px + qy - dz + Z(x,y,z) \end{cases}, \qquad \text{(IV.1)}$$

where $d > 0$ and X,Y,Z analytic and $\in o(|x|+|y|+|z|)$ $(x,y,z) \to (0,0,0)$.

Define $X_0(x,y)$ by $X(x,y,0)$ etc and $x'(x,y,z)$ by $X(x,y,z) - X_0(x,y)$ etc. with (IV.1) is associated a reduced system

$$\begin{cases} \dot{x} = -\lambda y + X_0(x,y) \\ \dot{y} = \lambda x + Y_0(x,y) \end{cases}. \qquad \text{(IV.2)}$$

Now we introduce the following hypotheses regarding (IV.1). In theorem 5 below we shall see that these hypotheses have important consequences. Later in this chapter we shall investigate methods by which the equations (IV.1) can be transferred into a form satisfying (H5), (H6) and (H7).

(H5) $\exists N \in \mathbb{N}$ such that G_{2N} (IV.2) $\neq 0$

(H6) $p = q = 0$

(H7) $z_0(x,y) \in O((|x| + |y|)^{2N-1})$.

<u>THEOREM 5</u>. *if* (IV.1) *satisfies* (H5) *through* (H7) *then there exists a function* $V(x,y,z)$ *such that in the neighbourhood of the origin* $V > 0$ *and* $\dot{V}_{(IV.1)}$ *is definite with the same sign as* $G_{2N(IV.2)}$.

<u>PROOF</u>. Without loss of generality we assume that $G_{2N} > 0$. (H5) $\Rightarrow \exists V_1(x,y)$ such that $\dot{V}_{1,(IV.2)} = G_{2N}(x^2+y^2)^N + \phi_0(x,y) \sum_{i+j\geq 2N} c^0_{ij} x^i j^i$ where ϕ_0 is analytic and $\phi_0(0,0) = 0$.

If $V_2(x,y,z) = V_1(x,y) - \frac{1}{2d} z^2$ then

$$\frac{dV_2}{dt} = \frac{\partial V_1}{\partial x} (-\lambda y + X^0(x,y) + X^1(x,y,z)) +$$

$$\frac{\partial V_1}{\partial y} (\lambda x + Y^0(x,y) + Y^1(x,y,z)) +$$

$$- \frac{1}{dz} (-dz + Z^0(x,y) + Z^1(x,y,z)) =$$

$$\{G_{2N}(x^2+y^2)^N + \phi_0(x,y) \sum_{i+j\geq 2N} c^0_{ij} x^i y^j + z^2\} +$$

$$\{\frac{\partial V_1}{\partial x} x^1(X,y,z) + \frac{\partial V_1}{\partial y} Y^1(x,y,z) - \frac{1}{d} z(Z^0(x,y) + Z^1(z,y,z))\}.$$

It is possible to represent all terms in $z \geq 2$ except z^2 as a function $\psi(x,y,z) z^2$ where ψ is analytic and $\psi(0,0,0) = 0$. It is also possible to represent all the terms linear in z and of order $\geq$ 2N in x,y as a function $\phi_1(x,y,z)\sum_{i+j\geq 2N} c^1_{ij} x^i y^j$ where ϕ_1 is analytic and $\phi_1(0,0,0) = 0$. Therefore

$$\frac{dV_2}{dt} = G_{2N}(x^2+y^2)^N + \phi_1(x,y,z) \sum_{i+j\geq 2N} c^1_{ij} x^i y^j + \psi(x,y,z) z^2 +$$

$$\sum_{k=2}^{2N-1} P_k(x,y,z) + z^2$$

where P_k is linear in z and of order k in x,y. The definiteness of $\dot{V}_2$ (VI.1) is disturbed by the presence of the terms P_k. By adding suitable terms to V_2 the terms P_k will be removed.

Introduce for

$$P_k = u_1 zx^k + u_2 zx^{k-1} y + \ldots + u_{k+1} zy^k$$

a term

$$Q_k = v_1 zx^k + v_2 zx^{k-1} y + \ldots + v_{k+1} zy^k$$

beginning at k = 2 and ending at k = 2N-1. Then we have

$$\dot{Q}_{k(IV.1)} = f_1 z x^k + f_2 z x^{k-1} y + \ldots + f_{k+1} z y^k +$$

$$\phi_k(x,y,z) \sum_{i+j\geq 2N} c_{ij}^k x^i y^j + \psi_k(x,y,z) z^2 + \sum_{j=k+1}^{2N-1} P_j^k(x,y,z)$$

where (i) $\phi_k(x,y,z) \sum_{i,j\leq 2N} c_{ij}^k x^i y^j$ represents the terms linear in z and of order $\geq 2N$ in x,y; ϕ_k analytic and $\phi_k(0,0,0) = 0$.

(ii) ψ_k is analytic and $\psi_k(0,0,0) = 0$

(iii) $P_j^k(x,y,z)$ represent the terms linear in z and of order j in x,y.

(iv) $f_j = -dv_j - \lambda(k-j+2)v_{j-1} + \lambda v_{j+1}$ $(j = 1,2,\ldots,k+1)$ where $v_{k+2} = v_0 = 0$.

Therefore we want to solve the system

$$-dv_j - \lambda(k-j+2)v_{j-1} + \lambda j v_{j+1} = -u_j \quad (j = 1,2,\ldots,k+1) \qquad \text{(IV.3)}$$

for $v_1, v_2, \ldots, v_{k+1}$ in terms of $u_1, u_2, \ldots, u_{k+1}$. Corollary 1 of the appendix shows that this system is indeed solvable.

The following theorem shows that (H6) and (H7) don't put serious conditions on system (IV.1).

THEOREM 6. *Suppose that* (IV.1) *satisfies* (H5) *and, in that connection let* N *denote the smallest positive integer k for which* $G_{2k(IV.2)} \neq 0$. *Then there exists a coordinate transformation*

$$x \to \tilde{x} \qquad y \to \tilde{y} \qquad z \to \tilde{z},$$

valid near the origin such that in the new coordinates (IV.1) *satisfies* (H5), (H6), (H7) *and such that in the new coordinates* N *and* $G_{2N(IV.2)}$ *have the same values as in the old coordinates.*

PROOF. Apply a coordinate transformation of the following form:

$$\xi = z - (v_1(x,y) + v_2(x,y) + \ldots + v_{2N}(x,y)) \qquad \text{(IV.4)}$$

where the v_i's stand for homogeneous polynomials of degree i in x,y. Substitute (IV.4) in (IV.1) using $v(x,y) = \sum_{j=1}^{2N-1} v_j(x,y)$.

This gives

$$\left\{\begin{aligned}\dot{x} &= \quad\quad -\lambda y + X(x,y,\xi+v(x,y)) \\ \dot{y} &= \lambda x \quad\quad + Y(x,y,\xi+v(x,y)) \\ \dot{w} &= px+qy-d\xi + Z(x,y,\xi+v(x,y)) + \\ &\quad - [\frac{\partial v}{\partial x}(-\lambda y+X(x,y,\xi+v(x,y))] \\ &\quad - [\frac{\partial v}{\partial y}(\lambda x+Y(x,y,\xi+v(x,y))] \\ &\quad - dv(x,y)\end{aligned}\right\}, \tag{IV.5}$$

Denote $W(x,y,0)$ by $W_0(x,y)$. Clearly we have

$$\begin{aligned}W_0(x,y) &= px + qy - dv(x,y) + Z(x,y,v(x,y) \\ &\quad - [\frac{\partial v}{\partial x}(-\lambda y + X(x,y,v(x,y))] \\ &\quad - [\frac{\partial v}{\partial y}(\lambda x + Y(x,y,v(x,y))].\end{aligned} \tag{IV.6}$$

With the same techniques with which we determined the Q_k's in the previous theorem, it can be proved that one can choose $v_1(x,y)$ through $v_{2N-1}(x,y)$ such that in W_0 the terms in x and y with order up to 2N vanish. In that case (IV.5) satisfies the requirements of theorem 6, and hence our proof is complete.

Combining theorems 5 and 6 we have proved:

<u>THEOREM 7</u>. *If* (IV.1) *satisfies* (H5) *then there exists a function* $V(x,y,z)$ *such that in a neighbourhood of the origin* $V > 0$ *and* $\dot{V}_{(IV.1)}$ *is definite.*

V. A STABILITY FORMULA FOR THREE DIMENSIONAL SYSTEMS

Let (I.17) satisfy (H_1) through (H_4), (H_5) with N = 2 and (H_6). Let $x \in \mathbb{R}^3$. At $\mu = 0$ this system can be written as

$$\left\{\begin{aligned}\dot{x} &= -\lambda y + X(x,y,z) \\ \dot{y} &= \lambda x + Y(x,y,z) \\ \dot{z} & \quad -dz + Z(x,y,z)\end{aligned}\right\} \tag{V.1}$$

where

$$X(x,y,z) = \sum_{\substack{k,\ell\geq 0\\ k+\ell\geq 1}} C_{kl}(z)x^k y^l, \qquad X \text{ analytic}$$

$$Y(x,y,z) = \sum_{\substack{k,\ell\geq 0\\ k+\ell\geq 1}} D_{kl}(z)x^k y^l, \qquad Y \text{ analytic.}$$

Furthermore we represent $C_{k1}(z)$ and $D_{k1}(z)$ as power series:

$$C_{kl}(z) = C_{k\ell}^{(0)} + C_{k\ell}^{(1)}z + \dots$$

$$D_{kl}(z) = D_{k\ell}^{(0)} + D_{k\ell}^{(1)}z + \dots$$

Because X and Y do not contain terms of order one we have:

$$C_{10}^{(0)} = C_{01}^{(0)} = D_{10}^{(0)} = D_{01}^{(0)} = 0.$$

In aid of a short notation we define:

$$C_{10}^{(1)} := \alpha \qquad C_{01}^{(1)} := \beta$$

$$D_{10}^{(1)} := \gamma \qquad D_{01}^{(1)} := \delta.$$

Finally we represent Z as the following power series

$$Z(x,y,z) = J_{20}x^2 + J_{11}xy + J_{02}y^2 + \sum_{l=1}^{\infty} E_l(x,y)z^l,$$

Z analytic and E_l contains terms in x,y of order ≥ 1.

In order to fulfill (H_7) we apply the coordinate transformation

$$\zeta = z - v(x,y),$$

where

$$v(x,y) = v_{20}x^2 + v_{11}xy + v_{02}y^2,$$

$$\dot{\xi} = \dot{z} - v_x\dot{x} - v_y\dot{y} =$$

$$- dz + (J_{20}x^2 + J_{11}x^y + J_{02}y^2 + \ldots) +$$

$$- (2v_{20}x + v_{11}y)(-\lambda y + X(x,y,z)) +$$

$$- (v_{11}x + 2v_{02y})(\ \lambda x + Y(x,y,z)) =$$

$$- d\xi - d(v_{20}x^2 + v_{11}xy + v_{02}y^2) + (J_{20}x^2 + J_{11}xy + J_{02}y^2 + \ldots) +$$

$$- (2v_{20}x + v_{11}y)(-\lambda y + X(x,y,\xi + v(x,y)) +$$

$$- (v_{11}x + 2v_{02}y)(\ \lambda x + Y(x,y,\xi + v(x,y)).$$

Removing the second order terms in x and y we obtain the following conditions:

$$\begin{cases} dv_{20} + \lambda v_{11} = J_{20} \\ 2\lambda v_{20} - dv_{11} - 2\lambda v_{02} = - J_{11} \\ \lambda v_{11} - dv_{02} = - J_{02} \end{cases} \qquad \text{(V.2)}$$

which yields the solution

$$\begin{cases} v_{20} = J_{20}\dfrac{d^2+2\lambda^2}{d^3+4\lambda^2 d} - J_{11}\dfrac{\lambda}{d^2+4\lambda^2} + J_{02}\dfrac{2\lambda^2}{d^3+4\lambda^2 d} \\ v_{11} = J_{20}\dfrac{2\lambda}{d^2+4\lambda^2} + J_{11}\dfrac{d}{d^2+4\lambda^2} - J_{02}\dfrac{2\lambda}{d^2+4\lambda^2} \\ v_{02} = J_{20}\dfrac{2\lambda^2}{d^3+4\lambda^2 d} + J_{11}\dfrac{\lambda}{d^2+4\lambda^2} + J_{02}\dfrac{d^2+2\lambda^2}{d^3+4\lambda^2 d} \end{cases}. \qquad \text{(V.3)}$$

According to theorem 6 we have to replace z by v(x,y) in the first two equations and the stability property of the origin for this reduced system is the same as for the original problem.

Theorem 5B guarantees that if the origin is stable, a stable periodic orbit bifurcates from the origin for $\mu > 0$, and if the origin is unstable a unstable periodic orbit bifurcates from the origin for $\mu < 0$.

The reduced system for (V.1) becomes

$$\begin{cases} \dot{x} = -\lambda y + \tilde{X}(x,y) \\ \dot{y} = \lambda x + \tilde{Y}(x,y) \end{cases}, \tag{V.4}$$

where

$$\tilde{X}(x,y) = X(x,y,v(x,y)) \qquad \tilde{X} \text{ analytic,}$$

$$\tilde{Y}(x,y) = Y(x,y,v(x,y)) \qquad \tilde{Y} \text{ analytic,}$$

we represent $\tilde{X}$ and $\tilde{Y}$ as power series

$$\tilde{X}(x,y) = \sum_{\substack{i+j\geq 2 \\ i,j\geq 0}} X_{ij}x^i y^j,$$

$$\tilde{Y}(x,y) = \sum_{\substack{i+j\geq 2 \\ i,j\geq 0}} Y_{ij}x^i y^j,$$

and express the relevant coefficients in terms of C_{ij}, D_{ij}, $\alpha,\beta,\gamma,\delta$ and v_{ij}.

$$X_{20} = C_{20}^{(0)} \qquad Y_{20} = D_{20}^{(0)}$$
$$X_{11} = C_{11}^{(0)} \qquad Y_{11} = D_{11}^{(0)}$$
$$X_{02} = C_{02}^{(0)} \qquad Y_{02} = D_{02}^{(0)}$$

$$X_{30} = C_{30}^{(0)} + \alpha\, v_{20}$$
$$X_{12} = C_{12}^{(0)} + \beta\, v_{11} + \alpha\, v_{02}$$
$$Y_{21} = D_{21}^{(0)} + \gamma\, v_{11} + \delta\, v_{20}$$
$$Y_{03} = D_{03}^{(0)} + \delta\, v_{02}.$$

Using (III.9) and (V.3) we have:

$$G_{4,(V.1)} = 1/4\{3C_{30} + C_{12} + D_{21} + 3D_{02} +$$
$$+ \frac{1}{\lambda}[2C_{02}D_{02} - 2C_{20}D_{20} + C_{11}(C_{02}+C_{20}) - D_{11}(D_{02}+D_{20})]\}$$

$$+ \frac{1}{2} J_{20} \left[\left(\frac{1/2d(\alpha-\delta)+\lambda(\beta+\gamma)}{d^2+4\lambda^2}\right) + \frac{\alpha+\delta}{d}\right] +$$

$$+ \frac{1}{2} J_{11} \left(\frac{1/2d(\beta+\gamma)+\lambda(\delta-\alpha)}{d^2+4\lambda^2}\right) +$$

$$- \frac{1}{2} J_{02} \left[\left(\frac{1/2d(\alpha-\delta)+\lambda(\beta+\gamma)}{d^2+4\lambda^2}\right) - \frac{\alpha+\delta}{d}\right]. \qquad \text{(V.5)}$$

VI. THE LORENZ EQUATIONS

The Lorenz equations describe the flow occurring in a layer of uniform depth when the temperature difference between the upper and lower surfaces is maintained at a constant value.

If x is proportional to the intensity of the convective motion and y is proportional to the temperature difference between the ascending and descending currents; similar signs of x and y denoting that the warm fluid is rising and the cold fluid is descending, and if z is proportional to the distortion of the vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundary, then the equations are given by

$$\left\{\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned}\right\}, \qquad \text{(VI.1)}$$

σ, the so called Prandtl number, $= \kappa^{-1}\nu$ where κ is the coefficient of thermal expansion and ν is the viscosity; r, the Raleigh number, is the bifurcation parameter

We assume that

$$r > 1, \quad \sigma > b+1, \quad b > 0.$$

The equilibrium points are given by

$$x_0 = y_0 = \pm\sqrt{b(r-1)}\ , \quad z_0 = r-1$$

$$x_0 = y_0 = z_0 = 0.$$

The origin is not of interest because it is a saddle point in the linearized system for all values of the parameters. We perform the transformation

$$x \to x - \sqrt{b(r-1)} \qquad y \to y - \sqrt{b(r-1)} \qquad z \to z - r + 1.$$

In these new coordinates (VI.1) transforms into

$$\left.\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = x - y - \sqrt{b(r-1)}\,z - xz \\ \dot{z} = \sqrt{b(r-1)}\,x + \sqrt{b(r-1)}\,y - bz + xy \end{cases}\right\} \qquad \text{(VI.2)}$$

So the linearized system becomes

$$\left.\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = x - y - \sqrt{b(r-1)}\,z \\ \dot{z} = \sqrt{b(r-1)}\,x + \sqrt{b(r-1)}\,y - bz \end{cases}\right\}. \qquad \text{(VI.3)}$$

Denote by M the matrix of the linearized system (VI.3). So M is given by

$$\begin{pmatrix} \sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}. \qquad \text{(VI.4)}$$

$$\mathbf{det}(\lambda I-M) = \lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2\sigma b(r-1) \qquad \text{(VI.5)}$$

(VI.5) has two purely imaginary zero's $\lambda_{1,2} = \pm i\omega_0$ iff the product of the coefficients of λ^2 and λ equals the constant term. This is the case iff

$$r_0 = \frac{\sigma(\sigma+b+3)}{\sigma-b-1} \qquad \text{(VI.6)}$$

and then

$$\omega_0 = \sqrt{\frac{2\sigma b(\sigma+1)}{\sigma-b-1}}\,.$$

In this case the third zero equals $-(\sigma+b+1)$. Denote $\sqrt{\frac{\sigma+b+1}{2\sigma}}$ by g, then $\sqrt{b(r_0-1)} = g\omega_0$. A simple calculation shows that $\lambda_1(r_0) > 0$.

An eigenvector of M with eigenvalue $i\omega_0$ is given by

$$\xi_1 = \text{col} \quad (\sigma,\ \sigma+i\omega_0,\ \frac{\omega_0}{g} - i(\frac{1+\sigma}{g})) \tag{VI.7}$$

and an eigenvector of M with eigenvalue $-(\sigma+b+1)$ is given by

$$\xi_2 = \text{col} \quad (-\sigma,\ b+1,\ \frac{g\omega_0(\sigma-b-1)}{\sigma+1}). \tag{VI.8}$$

We now apply a transformation in $\mathbb{R}^3$ defined by the transformation matrix

$$P := (\text{Re}\xi_1\ \text{Im}\xi_1\ \xi_2) =$$

$$\begin{pmatrix} \sigma & 0 & -\sigma \\ \sigma & \omega_0 & b+1 \\ \frac{\omega_0}{g} & \frac{-1-\sigma}{g} & \frac{g\omega_0(\sigma-b-1)}{\sigma+1} \end{pmatrix}. \tag{VI.9}$$

Then

$$\Delta := \det P = \sigma\sqrt{\frac{2\sigma}{\sigma+b+1}}\left[(\sigma+b+1)^2 + \frac{2\sigma b(\sigma+1)}{\sigma-b-1}\right], \tag{VI.10}$$

$$P^{-1} = \frac{1}{\Delta}\begin{pmatrix} a_{11} & \frac{\sigma(1+\sigma)}{g} & \omega_0\sigma \\ a_{21} & \frac{g\omega_0\sigma(\sigma-b-1)}{\sigma+1} + \frac{\omega_0\sigma}{g} & -\sigma(\sigma+b+1) \\ a_{31} & \frac{\sigma(1+\sigma)}{g} & \omega_0\sigma \end{pmatrix} \tag{VI.11}$$

The first column of P^{-1} is not explicitly given because this column won't enter into the formulas.

In our new coordinate system (VI.2) becomes:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = P^{-1}MP\begin{pmatrix}x\\ y\\ z\end{pmatrix} + P^{-1}\begin{pmatrix}0\\ -(\sigma x-\sigma z)(\frac{\omega_0}{g}x - \frac{1+\sigma}{g}y + \frac{g\omega_0(\sigma-b-1)}{\sigma+1}z)\\ (\sigma x-\sigma z)(\sigma x + \omega_0 y + (b+1)z)\end{pmatrix} =$$

$$\begin{pmatrix}0 & -\omega_0 & 0\\ \omega_0 & 0 & 0\\ 0 & 0 & -(\sigma+b+1)\end{pmatrix}\begin{pmatrix}x\\ y\\ x\end{pmatrix} + (G).$$

Define the following numbers

$$a := \frac{\sigma(\sigma+1)}{g}, \qquad b_1 := \frac{g\omega_0\sigma(\sigma-b-1)}{\sigma+1}, \qquad c := \frac{\omega_0\sigma}{g},$$

$$e := \sigma(b+1), \tag{VI.12}$$

then,

$$(G) = \frac{1}{\Delta}\begin{pmatrix}a & \omega_0\sigma\\ b_1+c & -\sigma^2-c\\ a & \omega_0\sigma\end{pmatrix}\begin{pmatrix}-cx^2 + axy + (c-b_1)xz - azy + b_1z^2\\ \sigma^2x^2 + \sigma\omega_0xy + (e-\sigma^2)xz - \sigma\omega_0 - ez^2\end{pmatrix} =$$

$$\frac{1}{\Delta}\begin{pmatrix}(-ac+\omega_0\sigma^3)x^2 + (a^2+\sigma^2\omega_0^2)xy + \{a(c-b_1) + \omega_0\sigma(e-\sigma^2)\}xz\\ + (-a^2-\sigma^2\omega_0^2)zy + (ab_1 - e\omega_0\sigma)z^2\\ \{-(b_1+c)c + (-\sigma^2-e)\sigma^2\}x^2 + \{a(b_1+c) + (-\sigma^2-e)\sigma\omega_0\}xy +\\ (c^2-b_1^2+\sigma^4-e^2)xz + \{-a(b_1+c) + \sigma\omega_0(e+\sigma^2)\}zy + \{b_1(b_1+c) + e(\sigma^2+c)\}z^2\\ (-ac+\omega_0\sigma^3)x^2 + (a^2+\sigma^2\omega_0^2)xy + \{a(c-b_1) + \omega_0\sigma(e-\sigma^2)\}xz\\ + (-a^2-\sigma^2\omega_0^2)zy + (ab_1 - e\omega_0\sigma)z^2\end{pmatrix}. \tag{VI.13}$$

In the notation of the formula (V.5) we have

$$C_{20} = \frac{1}{\Delta}(-ac+\omega_0\sigma^3)$$

$$C_{11} = \frac{1}{\Delta}(a^2+\sigma^2\omega_0^2)$$

$$C_{02} = 0$$

$$D_{20} = \frac{1}{\Delta}\{-(b_1+c)c + (-\sigma^2-e)\sigma^2\}$$

$$D_{11} = \frac{1}{\Delta}\{a(b_1+c) + (-\sigma^2-e)\sigma\omega_0\}$$

$$D_{02} = 0$$

$$J_{20} = C_{20},\ J_{11} = C_{11},\ J_{02} = C_{02}$$

$$\alpha = \frac{1}{\Delta}\{a(c-b_1) + \omega_0\sigma(e-\sigma^2)\}$$

$$\beta = \frac{1}{\Delta}(-a^2-\sigma^2\omega_0^2)$$

$$\gamma = \frac{1}{\Delta}(c^2-b_1^2+\sigma^4-e^2)$$

$$\delta = \frac{1}{\Delta}\{-a(b_1+c) + \sigma\omega_0(e+\sigma^2)\}$$

$$d = \sigma + b + 1.$$

Now we are ready to apply (V.5). The final result is

$$\begin{aligned}G_{4,(VI.1)} = \frac{1}{4\Delta^2\omega_0}\{&-2(-ac+\omega_0\sigma^3)(-(b_1+c)c - (\sigma^2+e)\sigma^2) + \\ &(a^2+\sigma^2\omega_0^2)(-ac+\omega_0\sigma^3) + \\ &[a(b_1+c) + (-\sigma^2-e)\sigma\omega_0][(b_1+c)c + (\sigma^2+e)\sigma^2]\} + \\ \frac{1}{2\Delta^2}(-ac+\omega_0\sigma^3)&\left\{\frac{\frac{1}{2}d(2ac-2\omega_0\sigma^3)+\omega_0(-a^2-\sigma^3\omega_0^2+c^2-b_1^2+\sigma^4-e^2)}{d^2+4\omega_0^2}\right. \\ &\left. + \frac{-2ab_1+2\omega_0\sigma e}{d}\right\} + \\ \frac{1}{2\Delta^2}(a^2+\sigma^2\omega_0^2)&\left\{\frac{\omega_0(2\omega_0\sigma^3-2ac)+\frac{1}{2}d(-a^2-\sigma^2\omega_0^2+c^2-b_1^2+\sigma^4-e^2)}{d^2+4\omega_0^2}\right\}.\end{aligned} \tag{VI.15}$$

APPENDIX

Set

$$A_{d,k+1} = \begin{pmatrix} -d & 1 & & & \\ -k & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & k \\ & & & -1 & -d \end{pmatrix} = \left(a_{ij}\right)_{i,j=1}^{k+1}$$

and

$$F_{k+1} \in C(\mathbb{R}, \mathbb{R}) \text{ defined by: } F_{k+1}(d) = \det A_{d,k+1}$$

where $d \in \mathbb{R}$ and $k \in \mathbb{N}$.

<u>LEMMA 1</u>.

$$\det A_{d,k+1} = (-1)^{k+1}\left[d^{k+1} + C_{k-1,k+1}d^{k-1} + \ldots + \begin{cases} C_{0,k+1} & \text{if } k+1 \text{ is even} \\ C_{1,k+1}d & \text{if } k+1 \text{ is odd} \end{cases}\right]$$

where

$$C_{m,k+1} = \sum_{i_\ell < i_{\ell+1} - 2} i_1(k+1-i_1)i_2(k+1-i_2)\ldots i_{n_m}(k+1-i_{n_m})$$

and

$$n_m = \frac{k+1-m}{2} .$$

<u>PROOF</u>.

$$\det A_{d,k+1} = \sum_{\substack{\{\sigma_1,\ldots,\sigma_n\} \in \\ \text{perm}\{1,\ldots,k+1\}}} (-1)^{\sigma} a_{1\sigma_1},\ldots,a_{k+1,\sigma_{k+1}}$$

where

$\sigma = 1$ for an odd permutation and

$\sigma = 0$ for an even permutation.

Let i be the smallest number such that $a_{i\sigma_i} \neq -d \Rightarrow a_{i\sigma_i} = a_{i,i+1} = i \Rightarrow a_{i+1\sigma_{i+1}} = a_{i+1,i} = -k-1+i$. This causes one permutation in {1...k+1}, which gives rise to one minus sign in the product $\prod_{j=1}^{k+1} a_{j\sigma_j}$. This proves the lemma.

<u>COROLLARY 1</u>. *If* $d \neq 0$ *then* $F_k(d) \neq 0$

<u>COROLLARY 2</u>. *If* $d = 0$ *and* k *is odd then* $F_k(0) \neq 0$

<u>COROLLARY 3</u>. *If* $d = 0$ *and* k *is even then* $F_k(0) = 0$ *and* $\dot{F}_k(0) \neq 0$.

We investigate (VI.15) for b fixed and $\sigma \to \infty$

$$\begin{aligned} \omega_0 &\sim \sqrt{2b} && (\sigma\to\infty) \\ g &\sim \tfrac{1}{2}\sqrt{2} && (\sigma\to\infty) \\ a &\sim \sqrt{2}\,\sigma^2 && (\sigma\to\infty) \\ b_1 &\sim \sqrt{b}\,\sigma^{3/2} && (\sigma\to\infty) \\ c &\sim 2\sqrt{b}\,\sigma^{3/2} && (\sigma\to\infty) \\ e &\sim b\sigma && (\sigma\to\infty) \\ d &\sim \sigma && (\sigma\to\infty). \end{aligned} \tag{VI.16}$$

Using (VI.16) we see that

$$G_{4,(VI.1)} = -\tfrac{1}{2}\sqrt{2}\,\sigma^4 + O(\sigma^3) \qquad (\sigma\to\infty). \tag{VI.17}$$

This result is in contradiction to the result of J.E. MARSDEN & M. McCRACKEN ([12], page 148).

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